Topology Qualifying Examination

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Instructions: Solve **four** out of the **six** problems. Even if you attempt more than four problems, indicate which problems you want graded.

You must justify your claims either by direct arguments or by referring to theorems you know.

Problem 1. (a) Describe the universal covering space of $T := S^1 \times S^1$.

(b) Let X be a space that is path-connected and locally path-connected, and $\pi_1(X)$ is a finite group. Is it true that every continuous function $f : X \longrightarrow T$ lifts to the universal covering space of T? Justify your answer with a proof or a counterexample.

(c) Prove that any continuous map $f : \mathbf{R}P^2 \times \mathbf{R}P^2 \longrightarrow T$ is null-homotopic. (*Hint:* Use part (b).)

Problem 2. (a) Consider the spaces $X = S^1 \vee S^1 \vee S^2$ and $V = S^1 \times S^1$. Show that $H_i(X; \mathbb{Z}) = H_i(V; \mathbb{Z})$, for all $i \geq 0$.

(b) Are the spaces X and V above homotopy equivalent? Justify your answer.

(c) Give an example of a 3-fold covering space of X. A picture with labels indicating the lifts of the three pieces in the wedge sum $S^1 \vee S^1 \vee S^2$ will suffice. Is your example a normal covering space? Justify your answer.

Problem 3. For $n > 1$, let D^n denote the n-dimensional unit disc. Show that D^n is not homeomorphic to D^m for any $m \neq n$.

Problem 4. Suppose that X is a finite CW-complex, and that $p : Y \longrightarrow X$ is an n-sheeted covering space of X.

(a) Give the definition of the Euler characteristic $\chi(X)$ of X.

(b) Suppose that Y is an n-sheeted covering space of X. Prove that Euler characteristics have the following relation: $\chi(Y) = n \cdot \chi(X)$. (*Hint:* Show that a CW-structure on X lifts to one on Y.)

(c) Are the maps $p_* : H_i(Y) \longrightarrow H_i(X)$ induced by the covering map injections for all $i \geq 0$? Prove your answer or justify it with a counterexample.

Problem 5. For $k \geq 1$, let T_k denote the 2-dimensional torus with k points removed.

(a) Calculate the fundamental group $\pi_1(T_k)$.

(b) Calculate the homology groups $H_i(T_k,\ \mathbb{Z})$, for all $i \geq 0$.

Problem 6. (a) Explain how to construct a CW complex with homology groups as below. Explain and justify your steps.

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & i = 1; \\ \mathbb{Z}/2\mathbb{Z}, & i = 2; \\ 0, & i \geq 3 \end{cases}$$

(b) Determine the cellular complex $(C_*(X),\ d)$ associated to the cell decomposition you constructed in (a).

(c) Compute the co-homology groups $H^i(X;\mathbb{Z})$, $i \geq 0$, of the space X you constructed in (a).